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Automorphisms of Certain Lie Algebras of Upper Triangular Matrices over a Commutative Ring

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Let $\mathbf{R}$ be an arbitrary commutative ring with identity. Denote by $\mathbf{t}$ the Lie algebra over $\mathbf{R}$ consisting of all upper triangular n by n matrices over $\mathbf{R}$ and let $\mathbf{b}$ be the Lie subalgebra of $\mathbf{t}$ consisting of all matrices of trace 0. The aim of this paper is to give an explicit description of the automorphism group of the Lie algebra $\mathbf{t}$, which extends a result given by Doković. In addition, we explicitly describe the automorphism group of the Lie algebra $\mathbf{b}$ when n is a unit of $\mathbf{R}$.

1. INTRODUCTION

Let $\mathbf{R}$ be a commutative ring with identity and $\mathbf{R}^*$ be the group of invertible elements of $\mathbf{R}$. Let $M_n(\mathbf{R})$ be the $\mathbf{R}$-algebra of n by n matrices over $\mathbf{R}$ that has a structure of a Lie algebra over $\mathbf{R}$ with the bracket operation $[x, y] = xy - yx$. We denote by $\mathbf{t}$ the solvable subalgebra of this Lie algebra consisting of all upper triangular matrices. When $n > 1$, let $\mathbf{b}$ be the Lie subalgebra of $\mathbf{t}$ consisting of all matrices of trace 0. Doković [1] has described the automorphism group of the Lie algebra $\mathbf{t}$ when $\mathbf{R}$ is a connected commutative ring. Introducing a kind of new automorphism of $\mathbf{t}$ and modifying the proof of the main theorem in [1], we find that the condition for the connectedness of $\mathbf{R}$ can be removed, and so we can give a complete description of the automorphism group of the Lie algebra $\mathbf{t}$ over an arbitrary commutative ring. As an application of the result, we will give an explicit description of the automorphism group of Lie algebra $\mathbf{b}$.

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2. PRELIMINARIES

Following the notations in [1], we denote by e the identity matrix in $M_n(\mathbf{R})$ and by e_{ij} the matrix in $M_n(\mathbf{R})$ whose sole nonzero entry is 1 in the (i,j) position. Let $\mathbf{L}$ denotes $\mathbf{t}$ or $\mathbf{b}$ and let $\mathrm{Aut}(\mathbf{L})$ be the automorphism group of the Lie algebra $\mathbf{L}$. Let $\mathbf{n}$ be the ideal of $\mathbf{L}$ consisting of all strictly upper triangular matrices. Let

$$\mathbf{n}_1 = \mathbf{n}, \qquad \mathbf{n}_2 = [\mathbf{n}, \mathbf{n}_1], \qquad \mathbf{n}_3 = [\mathbf{n}, \mathbf{n}_2], \qquad \cdots$$

be the lower central series of $\mathbf{n}$. Every $\mathbf{n}_k$ is an ideal of $\mathbf{L}$. These ideals are invariant under all $\varphi \in \mathrm{Aut}(\mathbf{L})$ except in the case $n = 2$, $\mathbf{L} = \mathbf{b}$ and 2 is a zero divisor of $\mathbf{R}$. Let $\mathbf{a}_k$ $(1 \le k < n)$ be the free $\mathbf{R}$-module with basis $\{e_{1j};\ k < j \le n\}$, which is also an ideal of $\mathbf{L}$. Let $\mathbf{Re}$ be the set $\{re;\ r \in \mathbf{R}\}$ of scalar matrices in $\mathbf{t}$. We denote by $Z(\mathbf{L})$ the center of the Lie algebra $\mathbf{L}$.

LEMMA 2.1. *If $n > 1$ and $n \in \mathbf{R}^*$, both $\mathbf{b}$ and $\mathbf{R}e$ are ideals of $\mathbf{t}$ and $\mathbf{t} = \mathbf{b} \oplus \mathbf{R}e$.*

Proof. It is clear that both $\mathbf{b}$ and $\mathbf{R}e$ are ideals of $\mathbf{t}$. For any $x \in \mathbf{t}$, let $u_1, \ldots, u_n$ be the diagonal entries of x. Put $u = n^{-1}(u_1 + \cdots + u_n)$. Then $x = (x - ue) + ue$, where $x - ue \in \mathbf{b}$ and $ue \in \mathbf{R}e$. This means that $\mathbf{t} = \mathbf{b} + \mathbf{R}e$. On the other hand, if $x = ue \in \mathbf{b}$, then $nu = 0$, which implies that $x = 0$. The proof is completed. ∎

LEMMA 2.2. *$Z(\mathbf{t}) = \mathbf{R}e$ and if $n \in \mathbf{R}^*$, $Z(\mathbf{b}) = 0$.*

Proof. The first assertion follows immediately from [1, Lemma 2]. For the second assertion, we assume that $x = \sum_{i \le j} u_{ij} e_{ij} \in Z(\mathbf{b})$. By Lemma 2.1, it is clear that x is also an element of $Z(\mathbf{t})$ and so, by the first assertion, $u_{ij} = 0$ for $i \ne j$ and $u_{ii} = u_{jj}$. Since the trace of x is zero and $n \in \mathbf{R}^*$, we have $x = 0$. ∎

3. THE STANDARD AUTOMORPHISMS OF $\mathbf{t}$

The standard automorphisms of $\mathbf{t}$ are as follows:

(A) Central Automorphisms

The map φ: $x \mapsto x + \chi(x)e$, for all $x \in \mathbf{t}$, where χ: $\mathbf{t} \to \mathbf{R}$ is a homomorphism of Lie algebras with $1 + \chi(e) \in \mathbf{R}^*$ and $\mathbf{R}$ is regarded as an abelian Lie algebra, is an endomorphism of the Lie algebra $\mathbf{t}$. Since the map φ has an inverse φ^{-1}: $x \mapsto x - \chi(x)(1 + \chi(e))^{-1}e$, for all $x \in \mathbf{t}$, φ is an automorphism of the Lie algebra $\mathbf{t}$, which is called a *central automorphism* of $\mathbf{t}$. Since $\varphi(\mathbf{n}) = \mathbf{n}$, $\chi(x) = 0$ for all $x \in \mathbf{n}$. Denote $\chi(e_{ii})$ by α_i.

Then φ: $e_{ij} \mapsto e_{ij} + \delta_{ij}\alpha_i e$. It is easy to see that $\varphi(e) = (1 + \alpha_1 + \cdots + \alpha_n)e$ and $1 + \alpha_1 + \cdots + \alpha_n \in \mathbf{R}^*$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$. We denote by φ_α the central automorphism φ. If $S = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n;\ 1 + \alpha_1 + \cdots + \alpha_n \in \mathbf{R}^*\}$, then the map assigning to each central automorphism φ_α of $\mathbf{t}$ an element α of S is a bijection from the set of all central automorphisms of $\mathbf{t}$ onto S. If both $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are in S, then $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i = \alpha_i + \beta_i(1 + \alpha_1 + \cdots + \alpha_n)$ is also in S and $\varphi_\alpha \circ \varphi_\beta = \varphi_\gamma$ (see [1]). It is clear that $\varphi_0 = 1$ and $(\varphi_\alpha)^{-1} = \varphi_\beta$, where $0 = (0, \ldots, 0)$ and $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_i = -\alpha_i(1 + \alpha_1 + \cdots + \alpha_n)^{-1}$. Let $G_0 = \{\varphi_\alpha;\ \alpha \in S\}$. Then G_0 is a subgroup of Aut($\mathbf{t}$).

(*Note*: In [1] G_0 denotes the following *inner automorphism group* of $\mathbf{t}$ and G_1 denotes the *central automorphism group* of $\mathbf{t}$).

(B) Inner Automorphisms

Denote by $\mathbf{t}^*$ the set of invertible matrices in $\mathbf{t}$. Let $a \in \mathbf{t}^*$. The map $\varphi_a{:}x \mapsto axa^{-1}$, for all $x \in \mathbf{t}$, is an *inner automorphism* of $\mathbf{t}$. Denote by G_1 the set of all inner automorphisms of $\mathbf{t}$. Then G_1 is a subgroup of Aut($\mathbf{t}$).

We know from [1] that the groups G_1 and G_0 commute elementwise and $G_1 \cap G_0 = 1$. In addition, an involution automorphism ω_0: $x \mapsto -rx'r$, where x' is the transpose of the matrix x and $r = e_{1n} + e_{2,n-1} + \cdots + e_{n-1,2} + e_{n1}$, has been given in [1]. For an arbitrary commutative ring $\mathbf{R}$ we introduce the third kind of standard automorphisms of $\mathbf{t}$, which includes ω_0, as follows.

(C) Graph Automorphisms

Let ε be an idempotent of $\mathbf{R}$. Then $1 - \varepsilon$ is also an idempotent and $\varepsilon(1 - \varepsilon) = 0$. We define a map ω_ε: $x \mapsto \varepsilon x + (1 - \varepsilon)\omega_0(x)$ for all $x \in \mathbf{t}$. It is easy to show that ω_ε is an endomorphism of the Lie algebra $\mathbf{t}$. It follows by direct calculation that if both ε and ε' are idempotents in $\mathbf{R}$, then $1 - (\varepsilon - \varepsilon')^2$ is also an idempotent and $\omega_{\varepsilon'} \circ \omega_\varepsilon = \omega_{1-(\varepsilon-\varepsilon')^2}$. This implies that $\omega_\varepsilon^2 = 1$ and ω_ε is an automorphism of $\mathbf{t}$. Referring to a symmetry of the Dynkin diagram of the complex simple Lie algebra A_{n-1}, we called ω_ε a *graph automorphism* of $\mathbf{t}$. It is clear that $\omega_1 = 1$ and ω_0 is just ω_0 in [1]. The set of all graph automorphisms of $\mathbf{t}$ is a subgroup of Aut($\mathbf{t}$), which is denoted by G_2.

LEMMA 3.1. (1) *If $a \in \mathbf{t}^*$, then $c = \varepsilon a + (1 - \varepsilon)b$ with $b = ra'^{-1}r$ is in $\mathbf{t}^*$ and $\omega_\varepsilon \circ \varphi_a \circ \omega_\varepsilon = \varphi_c$.*

(2) *If $\alpha = (\alpha_1, \ldots, \alpha_n) \in S$, let $\gamma_i = \varepsilon\alpha_i + (1 - \varepsilon)\alpha_{n+1-i}$. Then $\gamma = (\gamma_1, \ldots, \gamma_n) \in S$ and $\omega_\varepsilon \circ \varphi_\alpha \circ \omega_\varepsilon = \varphi_\gamma$.*

Proof. (1) It is easy to show that $c^{-1} = \varepsilon a^{-1} + (1-\varepsilon)b^{-1}$. Noting $\omega_0 \circ \varphi_a \circ \omega_0 = \varphi_b$, where $b = ra'^{-1}r$ [1, Proposition 4], for all $x \in \mathbf{t}$ we have

$$\begin{aligned}\omega_\varepsilon \circ \varphi_a \circ \omega_\varepsilon(x) &= \omega_\varepsilon(\varepsilon\varphi_a(x) + (1-\varepsilon)\varphi_a \circ \omega_0(x))\\ &= \varepsilon\varphi_a(x) + (1-\varepsilon)\varphi_b(x)\\ &= \varphi_c(x).\end{aligned}$$

(2) It is obvious that $\gamma \in S$. Since both $\omega_\varepsilon \circ \varphi_\alpha \circ \omega_\varepsilon$ and φ_γ act trivially on $\mathbf{n}$ and

$$\begin{aligned}&\omega_\varepsilon \circ \varphi_\alpha \circ \omega_\varepsilon(e_{ii})\\ &\quad= \omega_\varepsilon \circ \varphi_\alpha(\varepsilon e_{ii} - (1-\varepsilon)e_{n+1-i,n+1-i})\\ &\quad= \omega_\varepsilon(\varepsilon e_{ii} - (1-\varepsilon)e_{n+1-i,n+1-i} + (\varepsilon\alpha_i - (1-\varepsilon)\alpha_{n+1-i})e)\\ &\quad= e_{ii} + \gamma_i e,\end{aligned}$$

we obtain that $\omega_\varepsilon \circ \varphi_\alpha \circ \omega_\varepsilon = \varphi_\gamma$. ∎

LEMMA 3.2. (1) $G_1 = 1$ *and* $G_2 \subseteq G_0$ *for* $n = 1$.
(2) $G_2 \subseteq G_0 \times G_1$ *for* $n = 2$.
(3) $(G_0 \times G_1) \cap G_2 = 1$ *for* $n > 2$.

Proof. If $n = 1$, we have $\mathbf{t} = \mathbf{R}e$ and for any idempotent ε in $\mathbf{R}$, $\omega_\varepsilon(e) = (2\varepsilon - 1)e$. It is clear that $G_1 = 1$ and $\omega_\varepsilon = \varphi_{2\varepsilon-2} \in G_0$.

If $n = 2$, for any idempotent ε in $\mathbf{R}$, take $\alpha = (\varepsilon - 1, \varepsilon - 1)$ and $a = e_{11} + (2\varepsilon - 1)e_{22}$. A simple calculation shows that e_{11}, e_{12}, and e_{22} are all invariant under $\varphi_\alpha \circ \varphi_a \circ \omega_\varepsilon$. Thus $\varphi_\alpha \circ \varphi_a \circ \omega_\varepsilon = 1$ and so $\omega_\varepsilon = (\varphi_\alpha \circ \varphi_a)^{-1} \in G_0 \times G_1$.

If $n > 2$ and $\omega_\varepsilon \in G_0 \times G_1$ for some idemponent ε in $\mathbf{R}$, let $\omega_\varepsilon = \varphi_\alpha \circ \varphi_a$. Then $\omega_\varepsilon(e_{11}) = \varphi_\alpha \circ \varphi_a(e_{11}) = \varphi_\alpha(e_{11} + x) = e_{11} + \alpha_1 e + x$, where $x \in \mathbf{n}$. On the other hand, we have $\omega_\varepsilon(e_{11}) = \varepsilon e_{11} - (1-\varepsilon)e_{nn}$. Hence $e_{11} + \alpha_1 e + x = \varepsilon e_{11} - (1-\varepsilon)e_{nn}$. Since $n > 2$, we have $\alpha_1 = 0$ and $\varepsilon = 1$, and this means $\omega_\varepsilon = \omega_1 = 1$. ∎

4. THE AUTOMORPHISM GROUP OF t

THEOREM 4.1. *Let* $\mathbf{R}$ *be an arbitrary commutative ring with identity. Then*

(1) $\mathrm{Aut}(\mathbf{t}) = G_0$ *for* $n = 1$.

(2) $\mathrm{Aut}(\mathbf{t}) = G_0 \times G_1$ *for* $n = 2$.

(3) $\mathrm{Aut}(\mathbf{t}) = (G_0 \times G_1) \rtimes G_2$ *for* $n > 2$.

Proof. If $n = 1$, then $G_1 = 1$ and $G_2 \subseteq G_0$ by Lemma 3. It is obvious that $\mathrm{Aut}(\mathbf{t}) = G_0$.

From now on, we assume that $n > 1$. We will follow the steps taken by [1, Theorem 6]. Let $G = G_0 \times G_1$ if $n = 2$ and $G = (G_0 \times G_1) \rtimes G_2$ if $n > 2$. For any $\varphi \in \mathrm{Aut}(\mathbf{t})$ we will show that $\varphi \in G$. To do this we will repeatedly replace φ with $\psi \circ \varphi$, where ψ is a suitable element of G, until we obtain an element of G.

We first show that by repeated replacement we may obtain that $\varphi(e_{1k}) = e_{1k}$, $k = 2, \dots, n$, the same in [1].

Since $\varphi(\mathbf{n}_{n-1}) = \mathbf{n}_{n-1}$, we have $\varphi(e_{1n}) = \alpha e_{1n}$ with $\alpha \in \mathbf{R}^*$. Let $a = e + (\alpha - 1)e_{nn}$ and replace φ with $\varphi_a \circ \varphi$. Then we have $\varphi(e_{1n}) = e_{1n}$. If $n = 2$, our goal is achieved. Next we assume that $n > 2$. Since $\varphi(\mathbf{n}_{n-2}) = \mathbf{n}_{n-2}$,

$$\varphi(e_{1,n-1}) = \alpha e_{1,n-1} + \beta e_{1n} + \gamma e_{2n},$$

for some $\alpha, \beta, \gamma \in \mathbf{R}$. Since $\varphi(\mathbf{a}_{n-2})$ is an ideal of $\mathbf{t}$, we have

$$\alpha e_{1,n-1} + \beta e_{1n} = [e_{11}, \varphi(e_{1,n-1})] \in \varphi(\mathbf{a}_{n-2}).$$

Thus it follows that $\alpha e_{1,n-1}, \gamma e_{2n} \in \varphi(\mathbf{a}_{n-2})$ and so $\mathbf{R}e_{1n} \oplus \mathbf{R}\alpha e_{1,n-1} \oplus \mathbf{R}\gamma e_{2n}$ is contained in $\varphi(\mathbf{a}_{n-2})$. It is clear that

$$\mathbf{R}e_{1n} \oplus \mathbf{R}\alpha e_{1,n-1} \oplus \mathbf{R}\gamma e_{2n} = \varphi(\mathbf{a}_{n-2}).$$

Since $\mathbf{a}_{n-2}$ is a free $\mathbf{R}$-module of rank 2 and $\varphi(e_{1n}) = e_{1n}$, $\mathbf{R}\alpha e_{1,n-1} \oplus \mathbf{R}\gamma e_{2n}$ is a free $\mathbf{R}$-module of rank 1. That means $\mathbf{R} \cong \mathbf{R}\alpha e_{1,n-1} \oplus \mathbf{R}\gamma e_{2n}$. Denote by θ this isomorphism. Assume that

$$\theta(1) = u\alpha e_{1,n-1} + v\gamma e_{2n}, \qquad \text{for some } u, v \in \mathbf{R},$$

$$\theta(\varepsilon) = u\alpha e_{1,n-1}, \qquad \text{for some } \varepsilon \in \mathbf{R},$$

$$\theta(\eta) = v\gamma e_{2n}, \qquad \text{for some } \eta \in \mathbf{R}.$$

Then $1 = \varepsilon + \eta$. Since $\theta(\varepsilon\eta) = \varepsilon\theta(\eta) = \varepsilon v\gamma e_{2n}$ and $\theta(\varepsilon\eta) = \eta\theta(\varepsilon) = \eta u\alpha e_{1,n-1}$, it follows that $\theta(\varepsilon\eta) \in \mathbf{R}\alpha e_{1,n-1} \cap \mathbf{R}\gamma e_{2,n} = 0$. This implies that $\varepsilon\eta = 0$ and both ε and η are idempotents in $\mathbf{R}$. If $\theta(\varepsilon') = \alpha e_{1,n-1}$, then $\theta(u\varepsilon') = u\alpha e_{1,n-1} = \theta(\varepsilon)$. Thus $u\varepsilon' = \varepsilon$. On the other hand,

$$\theta(\varepsilon') = \theta(\varepsilon'(\varepsilon + \eta)) = \varepsilon'\theta(\varepsilon) + \varepsilon'\theta(\eta) = \varepsilon' u\alpha e_{1,n-1} + \varepsilon' v\gamma e_{2n}.$$

So we have $\alpha e_{1,n-1} = \varepsilon' u\alpha e_{1,n-1} + \varepsilon' v\gamma e_{2n}$. Hence $\alpha = \varepsilon' u\alpha = \varepsilon\alpha$, and so $(1 - \varepsilon)\alpha = 0$. The same argument shows that $(1 - \eta)\gamma = 0$. Therefore there exists an idempotent ε in $\mathbf{R}$ such that

$$(1 - \varepsilon)\alpha = 0 \quad \text{and} \quad \varepsilon\gamma = 0.$$

Replacing φ with $\omega_\varepsilon \circ \varphi$, we have

$$\begin{aligned}
\varphi(e_{1n}) &= (2\varepsilon - 1)e_{1n},\\
\varphi(e_{1,n-1}) &= \varepsilon(\alpha e_{1,n-1} + \beta e_{1n} + \gamma e_{2n})\\
&\quad + (1-\varepsilon)\omega_0(\alpha e_{1,n-1} + \beta e_{1n} + \gamma e_{2n})\\
&= \varepsilon\alpha e_{1,n-1} + \varepsilon\beta e_{1n} - (1-\varepsilon)\beta e_{1n} - (1-\varepsilon)\gamma e_{1,n-1}\\
&= (\alpha - \gamma)e_{1,n-1} + (2\varepsilon - 1)\beta e_{1n}.
\end{aligned}$$

let $a = e + (2\varepsilon - 2)e_{nn}$. Replacing φ with $\varphi_a \circ \varphi$, we have

$$\begin{aligned}
\varphi(e_{1n}) &= e_{1n},\\
\varphi(e_{1,n-1}) &= \beta e_{1n} + \alpha e_{1,n-1}, \qquad \text{for some } \alpha, \beta \in \mathbf{R},\\
\varphi(e_{2n}) &= \beta' e_{1n} + \alpha' e_{1,n-1} + \gamma' e_{2n}, \qquad \text{for some } \alpha', \beta', \gamma' \in \mathbf{R}.
\end{aligned}$$

Since $\{e_{1n}, e_{1,n-1}, e_{2n}\}$ is a basis for the free $\mathbf{R}$-module $\mathbf{n}_{n-2}$ and $\varphi(\mathbf{n}_{n-2}) = \mathbf{n}_{n-2}$, it follows that $\{\varphi(e_{1n}), \varphi(e_{1,n-1}), \varphi(e_{2n})\}$ is also a basis for $\mathbf{n}_{n-2}$, and so

$$\det\begin{pmatrix} 1 & 0 & 0 \\ \beta & \alpha & 0 \\ \beta' & \alpha' & \gamma' \end{pmatrix} = \alpha\gamma' \in \mathbf{R}^*.$$

Hence $\alpha \in \mathbf{R}^*$. If $a = e + (\alpha^{-1} - 1)(e_{11} + e_{nn})$, by replacing φ with $\varphi_a \circ \varphi$ we have

$$\begin{aligned}
\varphi(e_{1n}) &= e_{1n},\\
\varphi(e_{1,n-1}) &= e_{1,n-1} + \beta e_{1n}.
\end{aligned}$$

Let $b = e + \beta e_{n-1,n}$. Replacing φ with $\varphi_b \circ \varphi$, we have $\varphi(e_{1k}) = e_{1k}$ for $k = n, n-1$. If $n = 3$, our goal is achieved.

In the case $n > 3$, we use induction. Assume that $\varphi(e_{1k}) = e_{1k}$ for $m < k \leq n$, where $2 \leq m \leq n-2$. Since $e_{1m} \in \mathbf{n}_{m-1}$ and $\varphi(\mathbf{n}_{m-1}) = \mathbf{n}_{m-1}$, we may assume that

$$\varphi(e_{1m}) = \sum_{j-i \geq m-1} \alpha_{ij} e_{ij}.$$

By the induction hypothesis, φ acts trivially on $\mathbf{a}_m$. Hence it is easy to see that by applying φ to $\mathbf{a}_{m-1} \cap \mathbf{n}_m = \mathbf{a}_m$, we have $\varphi(\mathbf{a}_{m-1}) \cap \mathbf{n}_m = \mathbf{a}_m$.

For $k = 2, 3, \dots, n - m$, we have

$$[e_{k,k+1}, \varphi(e_{1m})] = \sum_{j \ge m+k} \alpha_{k+1,j} e_{kj}$$
$$- \sum_{i \le k+1-m} \alpha_{ik} e_{i,k+1} \in \varphi(\mathbf{a}_{m-1}) \cap \mathbf{n}_m = \mathbf{a}_m.$$

It follows that $\alpha_{k+1,j} = 0$ for $j \ge m + k$. Thus we have

$$\varphi(e_{1m}) = \alpha_{1m} e_{1m} + \cdots + \alpha_{1n} e_{1n} + \alpha_{2,m+1} e_{2,m+1} + \cdots + \alpha_{2n} e_{2n}.$$

If $m + 1 \le i \le n - 1$, then

$$[\varphi(e_{1m}), e_{i,i+1}] = \alpha_{1i} e_{1,i+1} + \alpha_{2i} e_{2,i+1} \in \varphi(\mathbf{a}_{m-1}) \cap \mathbf{n}_m = \mathbf{a}_m.$$

This implies that $\alpha_{2i} = 0$ and so

$$\varphi(e_{1m}) = \alpha_{1m} e_{1m} + \cdots + \alpha_{1n} e_{1n} + \alpha_{2n} e_{2n}.$$

Since $n - 2 \ge m$, we have

$$[\varphi(e_{1m}), e_{nn}] = \alpha_{1n} e_{1n} + \alpha_{2n} e_{2n} \in \varphi(\mathbf{a}_{m-1}) \cap \mathbf{n}_m = \mathbf{a}_m.$$

It follows that $\alpha_{2n} = 0$ and

$$\varphi(e_{1m}) = \alpha_{1m} e_{1m} + \cdots + \alpha_{1n} e_{1n}.$$

Since φ acts as identity on $\mathbf{a}_m$, we have $\alpha_{1m} \in \mathbf{R}^*$. If $a = e + (\alpha_{1m} - 1)e_{mm}$, by replacing φ with $\varphi_a \circ \varphi$ we have

$$\varphi(e_{1k}) = e_{1k} \qquad \text{for } k > m,$$
$$\varphi(e_{1m}) = e_{1m} + \alpha_{1,m+1} e_{1,m+1} + \cdots + \alpha_{1n} e_{1n}.$$

Let $b = e + \alpha_{1,m+1} e_{m,m+1} + \cdots + \alpha_{1n} e_{mn}$. By replacing φ with $\varphi_b \circ \varphi$ we obtain that $\varphi(e_{1m}) = e_{1m}$. Thus by induction we have proved that we can get a φ such that $\varphi(e_{1k}) = e_{1k}$ for $k > 1$. For this φ, we now can prove that $\varphi \in G_0 \times G_1$ in exactly the same way as was Theorem 6 in [1]. We omit the repeated argument. ∎

5. THE AUTOMORPHISM GROUP OF b

We now use the result on the automorphism group of $\mathbf{t}$ to discuss the automorphisms of $\mathbf{b}$.

In this section assume that $n > 1$ and n is a unit of $\mathbf{R}$.

It is obvious that the restriction of an inner or graph automorphism of $\mathbf{t}$ to $\mathbf{b}$ is an automorphism of $\mathbf{b}$, which is also called an inner or graph automorphism. The restriction of a central automorphism φ_α of $\mathbf{t}$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ to $\mathbf{b}$ is an automorphism of $\mathbf{b}$ if and only if $\alpha_1 = \cdots = \alpha_n$. In that case the restriction is trivial since $Z(\mathbf{b}) = 0$. We denote by $\overline{\varphi}$ the restriction of an automorphism φ of $\mathbf{t}$ to $\mathbf{b}$. Also, we denote by $\overline{G}_1$ and $\overline{G}_2$ the inner and graph automorphism groups of $\mathbf{t}$, respectively.

It follows from Lemmas 3.1 and 3.2 that $\overline{G}_2 \subseteq \overline{G}_1$ if $n = 2$ and $\overline{G}_1 \cap \overline{G}_2 = 1$ and $\overline{G}_1$ is normalized by $\overline{G}_2$ if $n > 2$.

THEOREM 5.1. *Let $n > 1$ and let $\mathbf{R}$ be a commutative ring with identity in which $n \in \mathbf{R}^*$. Then*

(1) $\operatorname{Aut}(\mathbf{b}) = \overline{G}_1$ *for* $n = 2$.

(2) $\operatorname{Aut}(\mathbf{b}) = \overline{G}_1 \rtimes \overline{G}_2$ *for* $n > 2$.

Proof. Let $\overline{G} = \overline{G}_1$ if $n = 2$ and $\overline{G} = \overline{G}_1 \rtimes \overline{G}_2$ if $n > 2$. For any $\overline{\varphi} \in \operatorname{Aut}(\mathbf{b})$, by Lemma 2.2, $\overline{\varphi}$ can be lifted to an automorphism of $\mathbf{t}$, which acts trivially on $\mathbf{R}e$. We denote it by φ. By Theorem 4.1 we have $\varphi = \varphi_\alpha \circ \varphi_a$ if $n = 2$ and $\varphi = \varphi_\alpha \circ \varphi_a \circ \omega_\varepsilon$ if $n > 2$, where φ_α, φ_a, and ω_ε are central, inner, and graph automorphisms of $\mathbf{t}$, respectively. In both cases we have $\overline{\varphi}_\alpha = 1$. In the case $n = 2$, since $\varphi_a(e) = e$ and $\varphi(e) = e$, we have $\varphi_\alpha(e) = e$ and so $\varphi_\alpha = 1$ since $2 \in \mathbf{R}^*$. It follows that $\varphi = \varphi_a$ and $\overline{\varphi} = \overline{\varphi}_a \in \overline{G}_1$. In the case $n > 2$, since $\varphi_a \circ \omega_\varepsilon(e) = (2\varepsilon - 1)e$ and $\varphi(e) = e$, we have $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i = 2(\varepsilon - 1)/n$, $i = 1, \dots, n$. So $\overline{\varphi} = \overline{\varphi}_a \circ \overline{\omega}_\varepsilon \in \overline{G}_1 \rtimes \overline{G}_2$. The theorem is completed. ∎

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